Решить задачу Коши:

б $\quad u_t = \Delta u + (\cos t - 2\sin t)\, e^{x+y+z}, \quad t > 0, \quad (x,y,z) \in \mathbb{R}^3$

$\quad u|_{t=0} = \cos x \cos 2y \qquad (x,y,z) \in \mathbb{R}^3$

1) u_1: $\begin{cases} u_{1t} = \Delta u_1 + (\cos t - 2\sin t)\, e^{x+y+z} \\ u_1|_{t=0} = 0 \end{cases}$

2) u_2: $\begin{cases} u_{2t} = \Delta u_2 \\ u_2|_{t=0} = \cos x \cos 2y \end{cases}$

1) найд. частное реш-е

$\Delta\left(e^{x+y+z}\right) = e^{x+y+z} \Rightarrow$ собств. ф-я

$u_1 = f(t) \cdot e^{x+y+z}$

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$\Delta \left(e^{x+y+z} \right) = e^{x+y+z} \Rightarrow$ собств. ф-я

$u_1 = f(t) \cdot e^{x+y+z}$

$$f' = 3f + \cos t - 2\sin t$$

$$\lambda = 3 \rightarrow f_0 = C_1 e^{3t}$$

$$f_2 = A \sin t + B \cos t$$

$$A \cos t - B \sin t = 3A \sin t + 3B \cos t + \cos t - 2 \sin t$$

$\cos t \Big|$ $A = 3B + 1$ $\qquad -B = 9B + 1$

$-B = 3A - 2$ $\qquad B = -1/10, \; A = 7/10$

$$f(0) = 0 = C_1 - \frac{1}{10} = 0 \Rightarrow C_1 = 1/10$$

$$\boxed{u_1 = \left(\frac{1}{10} e^{3t} + \frac{7}{10} \sin t - \frac{1}{10} \cos t \right) e^{x+y+z}}$$

Ответ:

$$u = e^{-5t}\cos x \cos 2y + \left(\frac{1}{10}e^{3t} + \frac{7}{10}\sin t - \frac{1}{10}\cos t\right)e^{x+y+z}$$

$u_2 = u_{21}\cdot u_{22}$

$u_{21}:\ \begin{cases} u_{21t} = \Delta u_{21} \\ u_{21}\big|_{t=0} = \cos x \end{cases}$

$\Delta(\cos x) = -\cos x$

$u_{21} = f(t)\cdot\cos x$

$f' = -f \qquad f = C_1 e^{-t}$

$\lambda = -1 \qquad f(0) = 1 \Rightarrow C_1 = 1$

$u_{21} = e^{-t}\cos x$

$u_{22}:\ \begin{cases} u_{22t} = \Delta u_{22} \\ u_{22}\big|_{t=0} = \cos 2y \end{cases}$

$\Delta(\cos 2y) = -4\cos 2y$

$u_{22} = f(t)\cdot\cos 2y$

$f' = -4f \qquad f = C_1 e^{-4t} \to f = e^{-4t}$

$\lambda = -4 \qquad f(0) = 1 \qquad C_1 = 1$

$u_{22} = e^{-4t}\cos 2y$

$$u_2 = e^{-t}\cos x\cdot e^{-4t}\cos 2y = \boxed{e^{-5t}\cos x\cos 2y}$$

II.2 в Решите задачу Коши

$$\begin{cases} u_t = \Delta u + (x^2 + y^2 - 2z^2)\cos t, & t>0, \quad (x,y,z)\in\mathbb{R}^3 \\ u|_{t=0} = x\cos(x+y) \end{cases}$$

1) $$\begin{cases} u_{1t} = \Delta u_1 + (x^2+y^2-2z^2)\cos t, & t>0 \\ u|_{t=0} = 0 \end{cases} \quad (*)$$

2) $$\begin{cases} u_{2t} = \Delta u_2 \\ u_2|_{t=0} = x\cos(x+y) \end{cases}$$

1). $\Delta(x^2+y^2-2z^2) = 2+2-4=0$ – соб. ф-я

$$u_1 = f(t)\cdot(x^2+y^2-2z^2)$$

$$f'(x^2+y^2-2z^2) = f\cdot 0 + (x^2+y^2-2z^2)\cos t \quad |:(\ldots)\neq 0$$

$$\begin{cases} f' = \cos t \\ f(0)=0 \end{cases} \Rightarrow \begin{array}{l} f = \sin t + C \\ C = 0 \end{array} \Rightarrow u_1 = \sin t\,(x^2+y^2-2z^2)$$

2) $\begin{cases} u_{2t} = \Delta u_2 \\ u_2|_{t=0} = x\cos(x+y) \end{cases}$

I $\cos(x+y) = \cos x \cos y - \sin x \sin y$

$u_2|_{t=0} = x\cos x \cdot \cos y - x\sin x \cdot \sin y$

$u_2 = u_{21} + u_{22}$ $\quad u_{21} = u_{211} \cdot u_{212}$ $\quad u_{22} = u_{221} \cdot u_{222}$

$u_{211}|_{t=0} = x\cos x$ $\quad u_{212}|_{t=0} = \cos y$ $\quad u_{221}|_{t=0} = -x\cos x$ $\quad u_{222}|_{t=0} = \sin y$

II $\Delta\, x\cos(x+y) = -2\sin(x+y) - 2x\cos(x+y)$

$x\cos(x+y)$ — не с. ф., но $\sin(x+y)$ — с. ф.

$u_2 = f(t)\, x\cos(x+y) + g(t)\sin(x+y)$

$f(0) = 1 \quad g(0) = 0$

II

$\Delta\, x\cos(x+y) = -2\sin(x+y) - 2x\cos(x+y)$

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2) $\begin{cases} u_{2t} = \Delta u_2 \\ u_2\big|_{t=0} = x\cos(x+y) \end{cases}$

$$u_2 = f(t)\, x\cos(x+y) + g(t)\sin(x+y)$$

$$f' \cdot x\cos(x+y) + g'\sin(x+y) = f(t)\cdot(-2\sin(x+y) - 2x\cos(x+y)) + g(t)(-2\sin(x+y))$$

$$\begin{cases} f' = f(t)(-2) & (1) \\ g' = g(t)(-2) - 2f(t) & (2) \\ f(0) = 1 \\ g(0) = 0 \end{cases}$$

1). $\begin{cases} f' + 2f = 0 \\ f(0) = 1 \end{cases}$

$\lambda + 2 = 0$

$\lambda = -2$

$f(t) = e^{-2t}\cdot C$

$f(0) = e^0 \cdot C = 1$

$\boxed{C = 1}$

$\Rightarrow \boxed{f(t) = e^{-2t}}$

$$\begin{cases} g' = g(t)(-2) - 2e^{-2t} \\ g(0) = 0 \end{cases}$$

$P_k(t)e^{\mu t}$

$g_2 = Q_k(t)e^{\mu t} \cdot t^S$

S - кратность корня хар. ур-я, равного μ

резонанс!

$g_0 = e^{-2t} \cdot \tilde{C} \Rightarrow \boxed{\tilde{C} = 0}$

$S = 1$

$g_2 = A \cdot e^{-2t} \cdot t$

$\cancel{-2Ate^{-2t}} + Ae^{-2t} = \cancel{-2Ate^{-2t}} - 2e^{-2t}$

$A = -2 \Rightarrow g_2 = -2te^{-2t}$

$g = -2te^{-2t}$

$$\boxed{u_2 = x\cos(x+y)\,e^{-2t} + \sin(x+y)(-2)te^{-2t}}$$

$$\boxed{U = \sin t\,(x^2 + y^2 - 2z^2) + x\cos(x+y)e^{-2t} + \sin(x+y)(-2)te^{-2t}}$$

ОТВЕТ.

П 2е

Найдите при каждом $t>0$ значение $u(0,0,t)$, решения задачи Коши

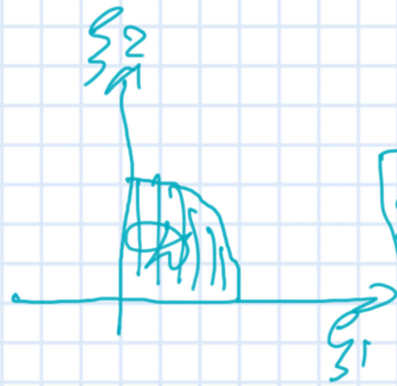
$$4u_t = \Delta u,\quad t>0,\quad (x,y)\in \mathbb{R}^2 \qquad a^2=\frac{1}{4}$$

$$u|_{t=0} = u_0(x,y), \text{ где}$$

$$u_0 = \begin{cases} x\cdot y\,(1-x^2-y^2), & \text{при } \{(x,y): x>0,\ y>0,\ x^2+y^2<1\} = D \\ 0, & \text{вне } D \end{cases}$$

$$u(0,0,t) = \left(\frac{1}{\sqrt{4\pi a^2 t}}\right)^2 \iint\limits_{-\infty}^{\infty} e^{-\frac{|\vec{\xi}-\vec{0}|^2}{4a^2 t}}\, u_0(\vec{\xi})\,d\xi_1\,d\xi_2 =$$

$$= \frac{1}{\pi t}\iint\limits_{D} e^{-|\vec{\xi}|^2/t}\,\xi_1\cdot\xi_2\,(1-\xi_1^2-\xi_2^2)\,d\xi_1\,d\xi_2 =$$



$$\boxed{\begin{aligned}\xi_1 &= r\cos\varphi \\ \xi_2 &= r\sin\varphi\end{aligned}}$$

$$= \frac{1}{\pi t}\int_0^1\int_0^{\pi/2} e^{-r^2/t}\, r\cdot r\cos\varphi\cdot r\sin\varphi\,(1-r^2)\,dr\,d\varphi$$

$$\int_0^{\pi/2}\cos\varphi\cdot\sin\varphi\,d\varphi = \frac{1}{2}\int_0^{\pi/2}\sin 2\varphi\,d\varphi = -\frac{\cos 2\varphi}{4}\Big|_0^{\pi/2} = \frac{1+1}{4} = \frac{1}{2}$$

$$= \frac{1}{\pi t}\,\frac{1}{2}\int_0^1 e^{-z^2/t}\, z^3(1-z^2)\,dz =$$

$$= \frac{1}{4\pi t}\int_0^1 e^{-z^2/t}\, z^2(1-z^2)\,dz^2 = \left| z^2 = s \right| =$$

$$= \frac{1}{4\pi t}\int_0^1 e^{-s/t}(s-s^2)\,ds = \underbrace{\frac{-t}{4\pi t}\, e^{-s/t}(s-s^2)\Big|_0^1}_{0} + \frac{1}{4\pi}\int_0^1 e^{-s/t}(1-2s)\,ds =$$

$$= \frac{-t}{4\pi}\, e^{-s/t}(1-2s)\Big|_0^1 + \frac{t}{4\pi}\int_0^1 e^{-s/t}\cdot(-2)\,ds =$$

$$= -\frac{t}{4\pi}\left(e^{-1/t}(1-2) - e^{0}\cdot 1\right) + -\frac{2t}{4\pi}(-t)\,e^{-s/t}\Big|_0^1 =$$

$$= \frac{t e^{-1/t}}{4\pi} + \frac{t}{4\pi} + \frac{t^2}{2\pi}\left(e^{-1/t} - 1\right) =$$

$$= \frac{1}{4\pi}\left[t\left(e^{-1/t}+1\right) + 2t^2\left(e^{-1/t}-1\right) \right]$$