УРАВНЕНИЕ ТЕПЛОПРОВОДНОСТИ - 2.

$$u_t = a^2 \Delta u + f(\vec{x}, t)$$

$(\vec{x}, t) \in G = \mathbb{R}^n \times (0; \infty)$,

$$f(\vec{x}, t) \in C(G)$$

Задача Коши

$$\begin{cases} u_t = a^2 \Delta u + f(\vec{x}, t), \\ u(\vec{x}, 0) = u_0(\vec{x}). \end{cases}$$

$u_0(\vec{x})$ - ● непрерывна и

● ограничена.

Уроев стр. 124.

ПРИНЦИП ДЮАМЕЛЯ

$$u(\vec{x},t)=\int\limits_0^t \frac{1}{\left(\sqrt{4\pi a^2(t-\tau)}\right)^n}\left[\int\limits_{\mathbb{R}^n} f(\vec{\xi},\tau)\exp\left(-\frac{\left|\vec{x}-\vec{\xi}\right|^2}{4a^2(t-\tau)}\right)d\vec{\xi}\right]d\tau + {}$$ ☹

$$+\frac{1}{\left(\sqrt{4\pi a^2 t}\right)^n}\int\limits_{\mathbb{R}^n} u_0(\vec{\xi})\exp\left(-\frac{\left|\vec{x}-\vec{\xi}\right|^2}{4a^2 t}\right)d\vec{\xi}$$

$$u(\vec{x},t)=u_p+v,$$

где

$$(u_p)_t=a^2\Delta u_p+f(\vec{x},t)$$

$$\begin{cases} v_t=a^2\Delta v, \\ v(\vec{x},0)=u_0(\vec{x})-u_p(\vec{x},0)=v_0(\vec{x}). \end{cases}$$

$$v(\vec{x},t)=\frac{1}{\left(\sqrt{4\pi a^2 t}\right)^n}\int\limits_{\mathbb{R}^n} v_0(\vec{\xi})\exp\left(-\frac{\left|\vec{x}-\vec{\xi}\right|^2}{4a^2t}\right)d\vec{\xi}$$

📖 👓 ✏ Уроев стр. 114 (пример 1), 117 (пример 2).

ПРИМЕР 1

$$\begin{cases} u_t = \Delta u + \left(x^2 + y^2 - 2z^2\right)\cos t, \\ u(\vec{x}, 0) = x\cos(x+y). \end{cases}$$

① $u(\vec{x}, t) = u_p + v$

$\boxed{u_p}$ $f(x,t) = \left(x^2 + y^2 - 2z^2\right)\cos t = g(t)\cdot h(x)$, $h(x) = x^2 + y^2 - 2z^2$,

$\Delta h(x) = \Delta\left(x^2 + y^2 - 2z^2\right) = 2 + 2 - 2\cdot 2 = 0$ → $h(x)$ - с.ф. Δ.

$u_p = w(t)\cdot\left(x^2 + y^2 - 2z^2\right)$

$w'(t) = \cos t$

желательно (но не обязательно) $w(0) = 0$ - НУ

$w(t) = \sin t + C$

$C = 0$ из **НУ**

$\underline{u_p = \left(x^2 + y^2 - 2z^2\right)\cdot \sin t}$

$\boxed{v}$ $\begin{cases} v_t = \Delta v, \\ v(\vec{x}, 0) = x\cos(x+y) \end{cases}$ - корректировки НУ не потребовалось. ☺

$$\Delta x\cos(x+y) = \frac{\partial^2}{\partial x^2} x\cos(x+y) + \frac{\partial^2}{\partial y^2} x\cos(x+y) =$$

$$= [0 - 2\sin(x+y) - x\cos(x+y)] - x\cos(x+y) =$$

$$= -2[\sin(x+y) + x\cos(x+y)]$$ - не с.ф. Δ. ☹

$$v = f(t)x\cos(x+y) + g(t)\sin(x+y).$$

$$f'x\cos(x+y) + g'\sin(x+y) = -2f\cdot[\sin(x+y) + x\cos(x+y)] - 2g\cdot\sin(x+y)$$

$$\begin{cases} f_t = -2f, \\ f(0) = 1. \end{cases}$$

$$\begin{cases} g_t = -2g - 2f, \\ g(0) = 0. \end{cases}$$

$\boxed{f}$ $f = Ce^{-2t}$. Из НУ $C = 1$, $f = e^{-2t}$.

$\boxed{g}$ $\begin{cases} g_t = -2g - 2e^{-2t}, \\ g(0) = 0. \end{cases}$ $g_o = Ce^{-2t}$, $g_p = ate^{-2t}$ - резонанс.

$ae^{-2t} - 2ate^{-2t} = -2ate^{-2t} - 2e^{-2t}$, $a = -2$, $g = Ce^{-2t} - 2te^{-2t}$. Из НУ $C = 0$, $g = -2te^{-2t}$.

$\underline{v = e^{-2t} x\cos(x+y) - 2te^{-2t}\sin(x+y)}$.

Ответ: $u(\vec{x}, t) = u_p + v =$

$= (x^2 + y^2 - 2z^2)\cdot \sin t + e^{-2t}\left[x\cos(x+y) - 2t\sin(x+y)\right]$ ❶

ПРИМЕР 2

$$\begin{cases} u_t = \Delta u + 27(t^2 - 1)\cos(x+y+z), \\ u(\vec{x}, 0) = (x - y + z)\sin z - (z-1)^3 e^{-(x-y)^2}. \end{cases}$$

② $u(\vec{x}, t) = u_p + v$

$\boxed{u_p}$ $f(x,t) = 27(t^2 - 1)\cos(x+y+z) = g(t)\cdot h(x), \quad h(x) = \cos(x+y+z),$

$\Delta h(x) = \Delta \cos(x+y+z) = -3\cos(x+y+z) =$ → $h(x)$ - с.ф. Δ.

$u_p = w(t)\cdot\cos(x+y+z)$

$$w'(t) = -3w + 27(t^2 - 1)$$

желательно (но не обязательно) $w(0) = 0$ - НУ

$$w(t) = Ce^{-3t} + at^2 + bt + c$$

$$2at + b = -3at^2 - 3bt - 3c + 27t^2 - 27$$

$$\begin{cases} 0 = -3a + 27, \\ 2a = -3b, \\ b = -3c - 27. \end{cases} \to \begin{cases} a = 9, \\ b = -6, \\ c = -7. \end{cases} \quad w(t) = Ce^{-3t} + 9t^2 - 6t - 7$$

$C = 7$ из **НУ**

$$\underline{u_p = \left(7e^{-3t} + (3t-1)^2 - 8\right)\cdot\cos(x+y+z)}$$

$\boxed{v}$ $\begin{cases} v_t = \Delta v, \\ v(\vec{x},0) = (x-y+z)\sin z - (z-1)^3 e^{-(x-y)^2}. \end{cases}$

- корректировки НУ не потребовалось. ☺

$$v = v_1(x,y,z,t) + v_2(z,t) + v_3(z,t)\cdot v_4(x,y,t)$$

$$\begin{cases} v_{1_t} = \Delta v_1, \\ v_1(\vec{x},0) = (x-y)\sin z. \end{cases}$$

$$\begin{cases} v_{2_t} = \Delta v_2, \\ v_2(\vec{x},0) = z\sin z. \end{cases}$$

$$\begin{cases} v_{3_t} = \Delta v_3, \\ v_3(\vec{x},0) = -(z-1)^3. \end{cases}$$

$$\begin{cases} v_{4_t} = \Delta v_4, \\ v_4(\vec{x},0) = e^{-(x-y)^2}. \end{cases}$$

$\boxed{v_1}$ $\begin{cases} v_{1_t} = \Delta v_1, \\ v_1(x,y,z,0) = (x-y)\sin z. \end{cases}$

$\Delta(x-y)\sin z = -(x-y)\sin z$ - с.ф. Δ. ☺

$v_1 = f(t)(x-y)\sin z$.

$\begin{cases} f_t = -f, \\ f(0) = 1. \end{cases}$

$f = Ce^{-t}$.

Из НУ $C = 1$, $f = e^{-t}$, а

$\underline{v_1 = e^{-t}(x-y)\sin z}$.

$\boxed{v_2}$ $\begin{cases} v_{2t} = \Delta v_2, \\ v_2(\vec{x}, 0) = z \sin z. \end{cases}$

$\Delta z \sin z = 2\cos z - z\sin z$ - не с.ф. Δ. ☹

$v_2 = f(t) z \sin z + g(t) \cos z$.

$f' z \sin z + g' \cos z = f \cdot (2\cos z - z \sin z) - g \cdot \cos z$

$\begin{cases} f_t = -f, \\ f(0) = 1. \end{cases}$

$\begin{cases} g_t = -g + 2f, \\ g(0) = 0. \end{cases}$

$\boxed{f}$ $f = Ce^{-t}$. Из НУ $C = 1$, $f = e^{-t}$.

$\boxed{g}$ $\begin{cases} g_t = -g + 2e^{-t}, \\ g(0) = 0. \end{cases}$ $g_o = Ce^{-t}$, $g_p = ate^{-t}$ - резонанс.

$ae^{-t} - ate^{-t} = -ate^{-t} + 2e^{-t}$, $a = 2$, $g = Ce^{-t} + 2te^{-t}$. Из НУ $C = 0$, $g = 2te^{-t}$.

$\underline{v_2 = e^{-t} z \sin z + 2te^{-t} \cos z}$.

$\boxed{v_3}$ $\begin{cases} v_{3_t} = \Delta v_3, \\ v_3(\vec{x}, 0) = -(z-1)^3. \end{cases}$

$\Delta\left[-(z-1)^3\right] = -6(z-1)$ - не с.ф. Δ. ☹

$\Delta^2\left[-(z-1)^3\right] = \Delta\left[-6(z-1)\right] = 0$ - с.ф. Δ^2. ☺

$v_3 = f(t)\left[-(z-1)^3\right] + g(t)\left[-6(z-1)\right].$

$$f'\left[-(z-1)^3\right] + g'\left[-6(z-1)\right] = f\left[-6(z-1)\right]$$

$\begin{cases} f_t = 0, \\ f(0) = 1. \end{cases}$

$\begin{cases} g_t = f, \\ g(0) = 0. \end{cases}$

$\boxed{f}$ $f = C$. Из НУ $C = 1,\ f = 1$.

$\boxed{g}$ $\begin{cases} g_t = 1, \\ g(0) = 0. \end{cases}$ $g_o = t + C$. Из НУ $C = 0,\ g = t$.

$\underline{v_3 = -(z-1)^3 - 6t(z-1)}.$

$\boxed{v_4}$ $\begin{cases} v_{4_t} = \Delta v_4, \\ v_4(\vec{x}, 0) = e^{-(x-y)^2}. \end{cases}$

Формула Пуассона $v_4(\vec{x}, t) = \dfrac{1}{\left(\sqrt{4\pi a^2 t}\right)^n} \displaystyle\int_{\mathbb{R}^3} v_{40}(\vec{\xi}) \exp\left(-\dfrac{\left|\vec{x} - \vec{\xi}\right|^2}{4a^2 t}\right) d\vec{\xi}$

Замена $\xi = x - y$

$$\Delta v_4 = \frac{\partial^2}{\partial x^2} v_4 + \frac{\partial^2}{\partial y^2} v_4 = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial \xi} v_4 \xi_x\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial \xi} v_4 \xi_y\right) =$$

$$= \frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \xi} v_4\right)\xi_x + \frac{\partial}{\partial \xi}\left(-\frac{\partial}{\partial \xi} v_4\right)\xi_y = 2\frac{\partial^2}{\partial \xi^2} \widetilde{v}_4.$$

$$\begin{cases} \widetilde{v}_{4_t} = 2\dfrac{\partial^2}{\partial \xi^2} \widetilde{v}_4, \\ \widetilde{v}_4(\xi, 0) = e^{-\xi^2}. \end{cases}$$

Вспоминаем, что $\boxed{v(\vec{x},t)=\dfrac{e^{-\frac{x^2b^2}{\left(1+4a^2b^2t\right)}}}{\sqrt{1+4a^2b^2t}}}$ решение ЗК $\begin{cases} v_t=a^2\Delta v, \\ v(\vec{x},0)=e^{-(bx)^2}. \end{cases}$

$$\tilde{v}_4(\xi,t)=\frac{e^{-\frac{\xi^2}{(1+8t)}}}{\sqrt{1+8t}}.$$

Обратная замена $\underline{v_4(x,y,t)=\dfrac{e^{-\frac{(x-y)^2}{(1+8t)}}}{\sqrt{1+8t}}}.$

Ответ: $u(\vec{x},t)=u_p+v_1(x,y,z,t)+v_2(z,t)+v_3(z,t)\cdot v_4(x,y,t)=$

$$=\left(7e^{-3t}+(3t-1)^2-8\right)\cdot\cos(x+y+z)+e^{-t}(x-y)\sin z+$$

$$+e^{-t}z\sin z+2te^{-t}\cos z-(z-1)\left[(z-1)^2-6t\right]\frac{e^{-\frac{\xi^2}{(1+8t)}}}{\sqrt{1+8t}}$$

❷