1(4) Решить задачу:

$$x^2 u_{xx} - 4y^2 u_{yy} + x u_x - 4y u_y = 0, x > 0, y > 0 ,$$

$$u|_{y=1} = 4x^2, \qquad u_y|_{y=1} = 2x^2 .$$

$x^2 u_{xx} - 4y^2 u_{yy} + x u_x - 4y u_y = 0$, $x>0, y>0$

$u|_{y=1} = 4x^2$, $u_y|_{y=1} = 2x^2$

① Уравнение характеристик

$x^2 (dy)^2 - 4y^2 (dx)^2 = 0 \qquad x\,dy = \pm 2y\,dx$

$\frac{dy}{y} = \pm 2 \frac{dx}{x} \qquad \frac{y}{x^2} = const, \quad yx^2 = const$

Замена $\boxed{\xi = x^2 y, \ \eta = \frac{x^2}{y}}$ (+1)

② таблица 1

$$\begin{array}{r|l}
x & u_x = 2xy\, u_\xi + \frac{2x}{y} u_\eta \\
-4y & u_y = x^2 u_\xi - \frac{x^2}{y^2} u_\eta \\
x^2 & u_{xx} = 4x^2 y^2 u_{\xi\xi} + 8x^2 u_{\xi\eta} + \frac{4x^2}{y^2} u_{\eta\eta} + 2y u_\xi + \frac{2}{y} u_\eta \\
-4y^2 & u_{yy} = x^4 u_{\xi\xi} - \frac{2x^4}{y^2} u_{\xi\eta} + \frac{x^4}{y^4} u_{\eta\eta} + \frac{2x^2}{y^3} u_\eta
\end{array}$$

$u_{xy} = \left(x^2 u_\xi - \frac{x^2}{y^2} u_\eta\right)_x = 2x u_\xi - \frac{2x}{y^2} u_\eta +$

$+ x^2 (u_\xi)_x - \frac{x^2}{y^2} (u_\eta)_x =$

$= 2x u_\xi - \frac{2x}{y^2} u_\eta + x^2 \left[2xy\, u_{\xi\xi} + \frac{2x}{y} u_{\xi\eta}\right] -$

$- \frac{x^2}{y^2} \left[2xy\, u_{\eta\xi} + \frac{2x}{y} u_{\eta\eta}\right]$

(3) Таблица (коэффициенты при)

$$
\begin{array}{l|l}
u_{\xi\xi} & x^2\cdot 4x^2y^2 - 4y^2\cdot x^4 = \\
u_{\eta\eta} & x^2\cdot \frac{4x^2}{y^2} - 4y^2\cdot \frac{x^4}{y^4} = \\
u_{\xi\eta} & x^2\cdot 8x^2 - 4y^2\left(-\frac{2x^4}{y^2}\right) = 16x^4 \\
u_{\xi} & x\cdot 2xy - 4y\cdot x^2 + x^2\cdot 2y = 0 \\
u_{\eta} & x\cdot \frac{2x}{y} - 4y\cdot\left(-\frac{x^2}{y^2}\right) + x^2\frac{2}{y} - 4y^2\cdot\frac{2x^2}{y^3} = 0
\end{array}
$$

$\cancel{16x^4}\, u_{\xi\eta} = 0$

$$\boxed{u_{\xi\eta} = 0}$$

$u = f(\xi) + g(\eta)$ f, g — дважды непр. дифф. ф-ии

$$\boxed{u = f(x^2y) + g\left(\frac{x^2}{y}\right)}$$ (+1)

(4) Решение ЗК:

$$u\big|_{y=1} = f(x^2) + g(x^2) = 4x^2$$

$$u_y\big|_{y=1} = \left[f'(x^2y)(x^2y)'_y\right]_{y=1} + \left[g'\left(\frac{x^2}{y}\right)\cdot\left(\frac{x^2}{y}\right)'_y\right]_{y=1} =$$

$$= x^2f'(x^2) - x^2g'(x^2) = 2x^2$$

замена $x^2 := \tau$

$$\begin{cases} f(\tau) + g(\tau) = 4\tau \\ f'(\tau) - g'(\tau) = 2 \end{cases} \qquad f'(\tau) + g'(\tau) = 4$$

$$2f'(\tau) = 6$$

$f'(r) = 3 \to f(r) = 3r + C$ +0.5

$g(r) = 4r - f(r) = r - C$ +0.5

$u = f(x^2 y) + g\left(\frac{x^2}{y}\right) =$

$= 3x^2 y + \cancel{C} + \frac{x^2}{y} - \cancel{C}$

Ответ: $\boxed{u = 3x^2 y + \frac{x^2}{y}}$ +1