5(4) Решить задачу Коши:

$$u_t = 4\Delta u + 7\sin t\cos(y-x), t>0, (x,y,z)\in\mathbb{R}^3$$

$$u\big|_{t=0} = y^4 e^{-(x-z)^2}, (x,y,z)\in\mathbb{R}^3.$$

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1) См. N 13.2 – приём здесь работает

$$u(x,y,z,t) = u_1(x,y,z,t) + u_2(y,t)\cdot u_3(x,z,t)$$

$$\begin{cases} u_{1t} = 4\Delta u_1 + \underbrace{7\sin t}_{f(t)}\,\underbrace{\cos(y-x)}_{g(x,y)} \\ u_1\big|_{t=0} = 0 \end{cases} \quad (\mathrm{I})$$

$$\begin{cases} u_{2t} = 4\Delta u_2 \\ u_2\big|_{t=0} = y^4 \end{cases} \quad (\mathrm{II})$$

$$\begin{cases} u_{3t} = 4\Delta u_3 \\ u_3\big|_{t=0} = e^{-(x-z)^2} \end{cases} \quad (\mathrm{III})$$

$$②\quad (\mathrm{I}) \quad \Delta g \quad \Delta\cos(x-y) = -2\cos(x-y) -$$

с. ф. Δ

$$\rightarrow u_1 = f(t)\cdot\cos(x-y)$$

$$f'\cos(x-y) = -4\cdot 2\cos(x-y)\cdot f(t) + 7\sin t\cos(x-y)$$

не терять!!!

$$\begin{cases} f' = -8f + 7\sin t \\ f(0) = 0 \end{cases}$$

хотим $u_1\big|_{t=0} = 0$

$$f = \underbrace{Ce^{-8t}}_{f_{oo}} + \underbrace{a\sin t + b\cos t}_{f_{част}}$$

$$a\cos t - b\sin t = -8a\sin t - 8b\cos t + 7\sin t$$

$$\begin{cases} a = -8b \\ -b = -8a + 7 \end{cases} \longrightarrow \begin{cases} a + 8b = 0 \\ 8a - b = 7 \end{cases} \quad a = -8b$$

$$-65b = 7 \qquad b = -\frac{7}{65} \qquad a = \frac{56}{65}$$

$$f = Ce^{-8t} + \frac{56}{65}\sin t - \frac{7}{65}\cos t \qquad C = \frac{7}{65}$$

$$\boxed{u_1(x,y,z,t) = \frac{7}{65}\left(e^{-8t} + 8\sin t - \cos t\right)\cos(y-x)}$$

+1

(II) $u_2\big|_{t=0} = y^4$

$$\Delta y^4 = 12y^2 \quad \Delta^2 y^4 = 24 \quad \Delta^3 y^4 = 0$$

$$u_2 = f(t)\cdot y^4 + g(t)\cdot 12y^2 + h(t)\cdot 24$$

$$\begin{cases} f' y^4 + g' 12y^2 + h' \cdot 24 = 4\left[f\cdot 12y^2 + g\cdot 24 + h\cdot 0\right] \\ f(0) y^4 + g(0) 12y^2 + h(0)\cdot 24 = y^4 \end{cases}$$

$$\begin{cases} f' = 0 \\ f(0) = 1 \end{cases} \qquad \begin{cases} g' = 4f = 4 \\ g(0) = 0 \end{cases} \qquad \begin{cases} h' = 4\cdot g = 16\cdot t \\ h(0) = 0 \end{cases}$$

$$f = 1 \qquad g = 4t + C,\ C = 0 \qquad h = 8\cdot t^2 + C$$

$C = 0$ из НУ

$$g = 4t \qquad h = 8t^2$$

$$\boxed{u_2(x,y,t) = y^4 + 48ty^2 + 192t^2}$$ (+1)

(III) $$\begin{cases} u_3(\cancel{\xi},t) = 4\Delta u_3(x,z,t) \\ u_3(x,z,t) = e^{-(x-z)^2} \end{cases}$$

Формула Пуассона

$$u_3 = \left(\frac{1}{\sqrt{4\pi a^2 t}}\right)^2 \iint\limits_{-\infty}^{\infty} e^{-(\xi-\eta)^2} e^{-\frac{|\vec{\xi}-\vec{x}|^2}{4a^2 t}}\, d\xi\, d\eta$$

$$\vec{\xi} = \begin{pmatrix}\xi\\ \eta\end{pmatrix} \quad \vec{x} = \begin{pmatrix}x\\ z\end{pmatrix}$$

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прием поворота СК: $\xi = x - z$ — замена
$\gamma = x + z$

$U_x = U_\xi \cdot 1 \qquad U_z = U_\xi(-1)$

$U_{xx} = U_{\xi\xi} \cdot 1^2 \qquad U_{zz} = U_{\xi\xi}(-1)^2 = U_{\xi\xi}$

$U_3(x,z,t) = V(\xi,t)$

$$\begin{cases} V_t = 4 \cdot 2 V_{\xi\xi} \\ V\big|_{t=0} = e^{-\xi^2} \end{cases}$$

$8\Delta V$ — одномерная задача

$a^2 = 8$

$$V = \frac{1}{\sqrt{4\pi \cdot 8t}} \int_{-\infty}^{\infty} e^{-s^2} e^{-\frac{(s-\xi)^2}{4 \cdot 8t}} ds =$$

ф-лу помним $\to$ *)

Не помним $\to$ честно интегрируем

$$= \frac{1}{\sqrt{4\pi 8t}} \int_{-\infty}^{\infty} \exp\left[\frac{-1}{32t}\left(\underbrace{32ts^2 + s^2} - 2s\xi + \xi^2\right)\right] ds =$$

$+ \frac{\xi^2}{1+32t} - \frac{\xi^2}{1+32t}$

дополним до полн. квадрата

$$= \frac{1}{\sqrt{4\pi 8t}} e^{-\frac{1}{32t}\left[\frac{(\cancel{1}+32t)\xi^2 - \cancel{\xi^2}}{1+32t}\right]} \cdot$$

$$\cdot \int_{-\infty}^{\infty} \exp\left[\frac{-1}{32t}\left(s\sqrt{32t+1} - \frac{\xi}{\sqrt{32t+1}}\right)^2\right] ds =$$

замена: $\frac{s\sqrt{\ldots} - \frac{\xi}{\sqrt{\ldots}}}{\sqrt{32t}} = \eta$

$$= \frac{1}{\sqrt{4\pi 8t}} e^{-\frac{\xi^2}{1+32t}} \int_{-\infty}^{\infty} e^{-\eta^2} d\eta \frac{\sqrt{32t}}{\sqrt{32t+1}} =$$

$\sqrt{\pi} = 2\int_0^{\infty} e^{-\eta^2} d\eta = \frac{\sqrt{\pi}}{2}$

*) $\quad \xi^2$

$$= \frac{1}{\sqrt{4\pi\, 8t}}\, e^{-\frac{\xi^2}{1+32t}} \int_{-\infty}^{\infty} e^{-\eta^2} d\eta \frac{\sqrt{32t}}{\sqrt{32t+1}} =$$

замена: $\frac{\ldots}{\sqrt{32t}} = \eta$

$\sqrt{\pi} = 2\int_0^{\infty} e^{-\eta^2} d\eta = \frac{\sqrt{\pi}}{2}$

$$*) \quad = \frac{e^{-\frac{\xi^2}{1+32t}}}{\sqrt{32t+1}}$$

+2 очка если ф-ла выведена

$$U_3 = \frac{e^{-\frac{(x-z)^2}{1+32t}}}{\sqrt{32t+1}}$$

+1 очко за готовую формулу

Ответ: $U = U_1 + U_2 \cdot U_3 =$

$$= \frac{7}{65}\left(e^{-8t} - \cos t - \sin t\right)\cos(y-x) +$$

$$+ \left(y^4 + 48t\, y^2 + 192t^2\right) \cdot \frac{\exp\left[-\frac{(x-z)^2}{1+32t}\right]}{\sqrt{1+32t}}$$