5.④ Решить задачу Коши: $3u_{tt} = \Delta u + 6e^t \sin(x+y+z), \qquad (x,y,z) \in \mathbb{R}^3, \quad t > 0;$

$$u|_{t=0} = xy^3, \qquad u_t|_{t=0} = (x^3 - 3xy^2)\operatorname{sh} z, \qquad (x,y,z) \in \mathbb{R}^3.$$

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5(4) Решить задачу Коши

$$3u_{tt} = \Delta u + 6e^t \sin(x+y+z), \quad \Big| :3$$

$(x,y,z) \in \mathbb{R}^3, \ t>0$

$u\big|_{t=0} = xy^3, \quad u_t\big|_{t=0} = (x^3 - 3xy^2)\operatorname{sh} z$

$(x,y,z) \in \mathbb{R}^3$

① $u(x,y,z,t) = u_1(\dots) + u_2(\dots) + u_3(x,y,z,t)$

$$\begin{cases} u_{1tt} = \frac{1}{3}\Delta u_1 + \boxed{2e^t}\boxed{\sin(x+y+z)} \\ u_1\big|_{t=0} = 0, \ u_{1t}\big|_{t=0} = 0 \end{cases} \qquad (\mathrm{I})$$

$f(t) \cdot g(x,z,y)$

$$\begin{cases} u_{2tt} = \frac{1}{3}\Delta u_2 \\ u_2\big|_{t=0} = xy^3, \ u_{2t}\big|_{t=0} = 0 \end{cases} \qquad (\mathrm{II})$$

$$\begin{cases} u_{3tt} = \frac{1}{3}\Delta u_3 \\ u_3\big|_{t=0} = 0 \quad u_{3t}\big|_{t=0} = (x^3 - 3xy^2)\operatorname{sh} z \end{cases} \qquad (\mathrm{III})$$

② (I) $\Delta g(x,y,z) = \Delta \sin(x+y+z) =$

$$= \frac{\partial^2}{\partial x^2}\sin(x+y+z) + \frac{\partial^2}{\partial y^2}\sin(x+y+z) +$$

$$+ \frac{\partial^2}{\partial z^2} = \sin(x+y+z) =$$

$$= -\sin(x+y+z)\cdot 1^2 + (-1)\sin(x+y+z)\cdot 1^2 +$$

$$+ (-1)\sin(x+y+z)\cdot 1^2 =$$

$$= -3\sin(x+y+z)$$ — с.ф. опер. Лапл.

$$u_1 = f(t)\sin(x+y+z)$$

$$f''\sin(x+y+z) = \frac{1}{3}\boxed{f(t)}(-3)\sin(x+y+z) + 2e^t\sin(x+y+z)$$

$$\begin{cases} f'' = -f + 2e^t \\ f(0) = 0, \; f'(0) = 0 \end{cases}$$ — ЗК для ОДУ

$f_{оо} = e^{\lambda t}$ хар. ур-е $\lambda^2 = -1$ $\lambda_{1,2} = \pm i$

$f_{оо}(t) = C_1\cos t + C_2 \sin t$

$f_2(t) = a e^t$ $\left(1 \neq \pm i\right.$ — не корень хар. ур-я$\left.\right)$

ур-я

2cos t sin(x+y+z) — нд. часть → резонанс

$f_2(t) = t\cdot a\cdot \cos t + t\cdot \underbrace{b}_{P_0(t)} \sin t$

$\cos t = \dfrac{e^{it}+e^{-it}}{2}$ → должен быть $\sin t$

$a e^t = -a e^t + 2e^t \qquad a = 1$

$f(t) = C_1 \cos t + C_2 \sin t + e^t$

НУ $\quad f(0) = C_1 + 1 = 0 \qquad f'(0) = C_2 + 1 = 0$

$$\boxed{u_1(x,y,z,t) = (e^t - \cos t - \sin t)\sin(x+y+z)}$$ +1

(II) $\quad xy^3 \qquad \Delta xy^3 = 6xy \qquad \Delta^2 xy^3 = 0$

$u_2(x,y,z,t) = f(t)\,xy^3 + g(t)\cdot 6xy$

$$\begin{cases} f'' xy^3 + g'' 6xy = \frac{1}{3}\left[f(t)\,6xy + g(t)\cdot 0\right] \\ f(0)\,xy^3 + g(0)\cdot 6xy = xy^3 \\ f'(0)\,xy^3 + g'(0)\cdot 6xy = 0 \end{cases}$$

xy^3 и $6xy$ — лин. незав.

$$\begin{cases} f''(t) = 0 \\ f(0) = 1 \\ f'(0) = 0 \end{cases} \qquad \begin{cases} g''(t) = \frac{1}{3} f(t) = \frac{1}{3} \\ g(0) = 0 \\ g'(0) = 0 \end{cases}$$

$f(t) = C_1 + C_2\cdot t \qquad g(t) = \frac{1}{3}\frac{t^2}{2} + d_1 + d_2\cdot t$

$C_1 = 1,\ C_2 = 0 \qquad d_1 = 0 \quad d_2 = 0$

$$\boxed{u_2(x,y,z,t) = xy^3 + t^2\cdot xy}$$ +1

(III) $(x^3-3xy^2)\cdot \operatorname{sh} z$ ⚠ Это ВУ, а не УТ

Захотели: $\cancel{U_3 = V_1\cdot V_2}$ ↓ $V_i = V_i(\underbrace{x,y}_{1},\underbrace{z}_{2},t)$ $i=1,2$

$$(V_1\cdot V_2)_{tt} = \underline{V_{1tt}} + 2V_{1t}\cdot V_{2t} + \underline{V_{2tt}} = \underline{a^2\Delta_{xy}V_1} + \underline{a^2\Delta_z V_2}$$

вот оно препятствие

$$\Delta\left[(x^3-3xy^2)\cdot \operatorname{sh}(z)\right] = \underbrace{\cancel{6x\operatorname{sh}z}}_{\frac{\partial^2}{\partial x^2}} - \underbrace{\cancel{6x\operatorname{sh}z}}_{\frac{\partial^2}{\partial y^2}} + \underbrace{(x^3-3xy^2)\operatorname{sh}z}_{\frac{\partial^2}{\partial z^2}} =$$

$$=(x^3-3xy^2)\operatorname{sh}z$$ — с. ф-я — уже приём рассмотрен

НО! ⟹ задумка автора могла быть другой (см. Колесникова С.И)

$$U_3(x,y,z,t) = \underbrace{(x^3-3xy^2)}_{\text{гармонич. ф-я}}\cdot V(z,t)$$: $\Delta(x^3-3xy^2)=0$

$$\begin{cases} V_{tt}(\dots) = \frac{1}{3}(\dots)V_{zz} \\ (\dots)V|_{t=0}=0,\ (\dots)V_t|_{t=0} = (\dots)\operatorname{sh}z \end{cases} \longrightarrow$$

$$\begin{cases} V_{tt}(z,t) = \frac{1}{3}V_{zz} \\ V|_{t=0} = 0 = V_0,\ V_t|_{t=0} = \operatorname{sh}z = V_1 \end{cases}$$ — одном. задача ⟹ можно воспольз.

по ф-лой Даламбера

$$V(z,t)=\frac{V_0\left(z+\frac{1}{\sqrt3}t\right)+V_0\left(z-\frac{1}{\sqrt3}t\right)}{2}+\frac{1}{2\frac{1}{\sqrt3}}\int_{z-\frac{t}{\sqrt3}}^{z+\frac{t}{\sqrt3}}V_1(\xi)\,d\xi=$$

$$=\frac{\sqrt3}{2}\int_{z-t/\sqrt3}^{z+t/\sqrt3}\operatorname{sh}\xi\,d\xi=$$

$$=\frac{\sqrt3}{2}\left[\operatorname{ch}\left(z+\frac{t}{\sqrt3}\right)-\operatorname{ch}\left(z-\frac{t}{\sqrt3}\right)\right]$$

$$\boxed{U_3=(x^3-3xy^2)\frac{\sqrt3}{2}\operatorname{sh}z\,\operatorname{sh}\frac{t}{\sqrt3}\cdot 2}$$

+2

Ответ: $U(x,y,z,t)=U_1+U_2+U_3=$

$$=(e^t-\cos t-\sin t)\sin(x+y+z)+$$

$$+xy^3+t^2xy+$$

$$+\frac{\sqrt3}{2}\left[\operatorname{ch}\left(z+\frac{t}{\sqrt3}\right)-\operatorname{ch}\left(z-\frac{t}{\sqrt3}\right)\right]$$