Функции Бесселя и их применение при решении задач для круглой мембраны. Метод Фурье

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Решить смешанную задачу для круга

$$u_t = \Delta u + y + e^{-\mu_{0,2}^2 t} J_0(\mu_{0,2} r), \quad x^2+y^2<1,\ t>0,$$
$$u|_{t=0} = J_2(\mu_{2,3} r)\cos\left(2\varphi - \frac{\pi}{6}\right), \quad x^2+y^2<1$$
$$u|_{r=1} = ty, \quad x^2+y^2=1,\ t>0,$$

где $r=\sqrt{x^2+y^2}$ и $\varphi=\varphi(x,y)$ — полярные координаты точки (x, y), $\varphi(0,0)=0$, $\mu_{n,i}$ — i-тый положительный нуль функции Бесселя $J_n(\rho)$

1. Замена искомой функции

$u = v + u^*$, где $u^*|_{r=1} = ty = t\sin\varphi$

$$u^* = \left[\begin{array}{l} t\sin\varphi \\ tr\sin\varphi \\ tr^{\alpha}\sin\varphi \\ \vdots \end{array}\right. \qquad \alpha = ?$$

$\Delta u^* = u^*_{rr} + \frac{1}{r}u^*_r + \frac{1}{r^2}u^*_{\varphi\varphi}$

$\longrightarrow$ $\alpha \geq 2$ для непрерывности решения

$\Delta r^{\alpha} t\sin\varphi = 0$ — экономичнее

$$\left(\alpha(\alpha-1) + \alpha - 1\right) r^{\alpha-2} t\sin\varphi = 0$$

$\alpha^2 = 1 \qquad \alpha = 1 < 2$ —

] $\alpha = 2$

$$t\Delta r^2\sin\varphi = t(2+2-1)\sin\varphi = 3t\sin\varphi$$

$u^* = r^2 t\sin\varphi \qquad u^*|_{t=0} = 0$

$$\begin{cases} v_t + r^2\sin\varphi = \Delta v + 3t\sin\varphi + r\sin\varphi + e^{-\mu_{0,2}^2 t}J_0(\mu_{0,2} r) \\ v|_{t=0} = J_2(\mu_{2,3} r)\cos\left(2\varphi - \frac{\pi}{6}\right) - 0 \\ v|_{r=1} = tr\sin\varphi|_{r=1} - tr^2\sin\varphi|_{r=1} = 0 \end{cases}$$

$$\begin{cases} v_t = \Delta v + (3t + r - r^2)\sin\varphi + e^{-\mu_{0,2}^2 t}J_0(\mu_{0,2} r) \\ v|_{t=0} = J_2(\mu_{2,3} r)\cos\left(2\varphi - \frac{\pi}{6}\right) \\ v|_{r=1} = 0 \end{cases}$$

II В силу линейности задачи

$$v = v_1 + v_2 + v_3 + v_4$$

v_1
$$\begin{cases} v_{1t} = \Delta v_1 + t \cdot 3\sin\varphi \\ v_1|_{t=0} = 0 \\ v_1|_{r=1} = 0 \end{cases}$$

$$v_1 = \sum_{k=1}^{\infty} J_1(\mu_{1,k} r)\sin\varphi \cdot T_k(t)$$

$$3 = \sum_{k=1}^{\infty} \alpha_k J_1(\mu_{1,k} r)\sin\varphi$$

$$\alpha_k = \frac{\int_0^1 3 J_1(\mu_{1,k} r)\cdot r\,dr}{\int_0^1 r J_1^2(\mu_{1,k} r)\,dr}$$

$$\begin{cases} T_k'(t) = -(\mu_{1,k})^2 T_k + t\alpha_k \\ T_k(0) = 0 \end{cases}$$

$$T_k = c e^{-(\mu_{1,k})^2 t} + \underbrace{at + b}_{\text{частное решение}}$$

$$a = -(\mu_{1,k})^2 at - (\mu_{1,k})^2 b + t\alpha_k$$

$$a = \frac{\alpha_k}{(\mu_{1,k})^2} \qquad b = -\frac{a}{(\mu_{1,k})^2} = -\frac{\alpha_k}{(\mu_{1,k})^4}$$

$$T_k = Ce^{-(\mu_{1,k})^2 t} + \frac{\alpha_k}{(\mu_{1,k})^2}\left(t - \frac{1}{(\mu_{1,k})^2}\right)$$

Из НУ: $T_k(0) = C - \frac{\alpha_k}{(\mu_{1,k})^4} = 0 \quad C = \frac{\alpha_k}{(\mu_{1,k})^4}$

$$T_k = \frac{\alpha_k}{(\mu_{1,k})^4}\left(e^{-(\mu_{1,k})^2 t} + (\mu_{1,k})^2 t - 1\right)$$

$$\boxed{v_1 = \sum_{k=1}^{\infty} J_1(\mu_{1,k} r)\sin\varphi \, \frac{e^{-(\mu_{1,k})^2 t} + (\mu_{1,k})^2 t - 1}{(\mu_{1,k})^4} \, \frac{\int_0^1 3 J_1(\mu_{1,k} r) r\,dr}{\int_0^1 r J_1^2(\mu_{1,k} r)\,dr}}$$

v_2
$$\begin{cases} v_{2t} = \Delta v_2 + (r - r^2)\sin\varphi \\ v_2|_{t=0} = 0 \\ v_2|_{r=1} = 0 \end{cases}$$

$$v_2 = \sum_{k=1}^{\infty} \beta_k J_1(\mu_{1,k} r)\sin\varphi \, T_k(t)$$

$$\beta_k = \frac{\int_0^1 (r - r^2) J_1(\mu_{1,k} r) r\,dr}{\int_0^1 r J_1^2(\mu_{1,k} r)\,dr}$$

$$\begin{cases} T_k' = -(\mu_{1,k})^2 T_k + \beta_k \\ T_k(0) = 0 \end{cases}$$

$$T_k = C e^{-(\mu_{1,k})^2 t} + \frac{\beta_k}{(\mu_{1,k})^2}$$

из НУ $C = -\frac{\beta_k}{(\mu_{1,k})^2}$

$$\boxed{v_2 = \sum_{k=1}^{\infty} J_1(\mu_{1,k} r)\sin\varphi \, \frac{1 - e^{-(\mu_{1,k})^2 t}}{(\mu_{1,k})^2} \, \frac{\int_0^1 (r^2 - r^3) J_1(\mu_{1,k} r)\,dr}{\int_0^1 r J_1^2(\mu_{1,k} r)\,dr}}$$

v_3
$$\begin{cases} v_{3t} = \Delta v_3 + e^{-(\mu_{0,2})^2 t} J_0(\mu_{0,2} r) \\ v_3|_{t=0} = 0 \\ v_3|_{r=1} = 0 \end{cases}$$

$$v_3 = J_0(\mu_{0,2} r)\, T(t)$$

$$\begin{cases} T' = -(\mu_{0,2})^2 T + e^{-(\mu_{0,2})^2 t} \\ T(0) = 0 \end{cases}$$

$$T(t) = C e^{-(\mu_{0,2})^2 t} + T_r$$ резонанс

$$T_r = A t e^{-(\mu_{0,2})^2 t}$$

$$A e^{-(\mu_{0,2})^2 t} \cancel{- A t (\mu_{0,2})^2 e^{-(\mu_{0,2})^2 t}} = \cancel{-(\mu_{0,2})^2 A t e^{-(\mu_{0,2})^2 t}} + e^{-(\mu_{0,2})^2 t}$$

$A = 1$

из НУ $C = 0$ $\qquad T = t e^{-(\mu_{0,2})^2 t}$

$$\boxed{v_3 = J_0(\mu_{0,2} r)\, t e^{-(\mu_{0,2})^2 t}}$$

v_4

$$\begin{cases} v_{4\,t} = \Delta v_4 \\ v_4\big|_{t=0} = J_2(\mu_{2,3} r)\cos(2\tilde{\varphi}), \quad \tilde{\varphi} = \varphi - \frac{\pi}{12} \\ v_4\big|_{r=1} = 0 \end{cases}$$

$$v_4 = J_2(\mu_{2,3} r)\cos 2\tilde{\varphi}\, T(t)$$

$$\begin{cases} T' = -(\mu_{2,3})^2 T \\ T(0) = 1 \end{cases} \qquad T = e^{-(\mu_{2,3})^2 t}$$

$$\boxed{v_4 = J_2(\mu_{2,3} r)\cos\left(2\varphi - \frac{\pi}{6}\right) e^{-(\mu_{2,3})^2 t}}$$

$$u = r^2 t \sin\varphi +$$

$$+ \sum_{k=1}^{\infty} J_1(\mu_{1,k} r)\sin\varphi \, \frac{e^{-(\mu_{1,k})^2 t} + (\mu_{1,k})^2 t - 1}{(\mu_{1,k})^4} \, \frac{\int_0^1 3 J_1(\mu_{1,k} r) r\, dr}{\int_0^1 r J_1^2(\mu_{1,k} r)\, dr} +$$

$$+ \sum_{k=1}^{\infty} J_1(\mu_{1,k} r)\sin\varphi \, \frac{1 - e^{-(\mu_{1,k})^2 t}}{(\mu_{1,k})^2} \, \frac{\int_0^1 (r^2 - r^3) J_1(\mu_{1,k} r)\, dr}{\int_0^1 r J_1^2(\mu_{1,k} r)\, dr} +$$

$$+ J_0(\mu_{0,2} r)\, t e^{-(\mu_{0,2})^2 t} +$$

$$+ J_2(\mu_{2,3} r)\cos\left(2\varphi - \frac{\pi}{6}\right) e^{-(\mu_{2,3})^2 t}$$